Modern approaches to quantum gravity

Solution 3 Fall 2025

1. The Penrose process

(a) The measure is trivially invariant under translations of t and ϕ (it has no t, ϕ dependence), thus K^μ and R^μ are Killing vectors.

(b) Using the metric,

$$0 = g_{\mu\nu} K^\mu K^\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \tag{1}$$

Recalling the definition of Δ,

$$K_\mu K^\mu = 0 \implies (r - GM)^2 = G^2 M^2 - a^2 \cos^2\theta \tag{2}$$

(c) Using that at the horizon, $\Delta(r_+) = 0$, $2GMr_+ = r_+^2 + a^2$, we write

$$g_{tt} = -\left(1 - \frac{2GMr}{\rho^2}\right) = -\frac{1}{\rho^2}(\Delta(r) - a^2\sin^2\theta) \to \frac{a^2 \sin^2\theta}{\rho^2} \tag{3}$$

$$g_{t\phi} \to -\frac{a(r_+^2 + a^2)\sin^2\theta}{\rho^2} \tag{4}$$

$$g_{\phi\phi} \to \frac{(r_+^2 + a^2)\sin^2\theta}{\rho^2} \tag{5}$$

Thus,

$$0 = \chi^\mu \chi_\mu = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi} = (\Omega_H(r_+^2 + a^2) - a)^2 \tag{6}$$

which implies the given result. Then, the condition $p_2^\mu \chi_\mu < 0$ reads

$$p_2^\mu K_\mu + \Omega_H p_2^\mu R_\mu = -E_2 + \Omega_H L_2 < 0 \implies L_2 < \frac{E_2}{\Omega_H} \tag{7}$$

Since we consider E_2 negative, L_2 is negative. Thus, it reduces to angular momentum of the black hole.

(d) The metric on the spatial slice Σ corresponding $dt = 0$, $dr = 0$ at $r = r_+$ reads,

$$ds_\Sigma^2 = (r_+^2 + a^2\cos^2\theta)d\theta^2 + \left[\frac{(r_+^2 + a^2)^2 \sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right] d\phi^2 \tag{8}$$

The determinant of this metric is

$$|\gamma| = (r_+^2 + a^2)^2 \sin^2\theta \tag{9}$$

Thus, one finds

$$A = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\phi \sqrt{|\gamma|} = 4\pi(r_+^2 + a^2) \tag{10}$$

(e) Plugging $r_+ = GM + \sqrt{G^2M^2 - a^2}$ and using $a = J/M$, one finds

$$A = 8\pi G^2 \left(M^2 + \sqrt{M^4 - (J/G)^2} \right) \tag{11}$$

Taking the variations with respect to M and J,

$$\delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial J}\delta J \tag{12}$$

one obtains

$$\delta A = \frac{8\pi G a}{\sqrt{G^2M^2 - a^2}}(\Omega_H^{-1}\delta M - \delta J) \tag{13}$$

In the process we considered, after waiting a long time for the p_1 particle to fly to infinity, we can describe the physics by a new Kerr black hole, with its momentum and the black hole mass changed by

$$\delta M = E_2 \quad \delta J = L_2 \tag{14}$$

However, we saw that the amount of energy we can extract is not arbitrary and has to satisfy the bound, which implies $\Omega_H^{-1}\delta M - \delta J > 0$. This precisely ensures that

$$\delta A > 0 \tag{15}$$

as required by the second law.

(f) The key of the process we discussed is that we were allowed to reduce the mass of the black hole, provided that we also reduce the rotation of the black hole. If we want to minimize the mass of the final black hole, we should start a process where we keep A fixed, and we reduce $J \to 0$. In this case we would have that the final mass, called the irreducible mass $M_{\text{irr}} = M_{\text{irr}}(M, J)$, is

$$8\pi G^2 \left(M^2 + \sqrt{M^4 - (J/G)^2} \right) = 16\pi G^2 M_{\text{irr}}^2 \tag{16}$$

In this case, by energy conservation

$$M + E_0 = M_{\text{irr}} + E_1 \tag{17}$$

we would have been able to extract

$$E_1 - E_0 = M - M_{\text{irr}} \tag{18}$$

2. Quantum field theory in Rindler coordinates

(a) Using $\sqrt{-g}\nabla_a V^a = \partial_a(\sqrt{-g}V^a)$ and using that modes with frequency ω will have the form $\Phi(t,r) = e^{-i\omega t}\phi(r)$ we have, as a remaining equation for ϕ,

$$0 = \nabla^a\nabla_a\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = \frac{1}{r^2}\left(r\partial_r(r\partial_r\phi) + \frac{\omega^2}{\kappa^2} \right) e^{-i\omega t}\,. \tag{19}$$

The solutions are as given in the exercise, $\phi \propto r^{\pm i|\omega|/\kappa} \equiv r^{iP}$. To analyze in which direction the wave propagates, one needs to check, when t increases, what x has to do in order for the phase to remain constant. When $P > 0$, if t increase, one has to increase x in order to follow the phase, thus it is right-moving. Similarly, when $P < 0$ and t increase, x needs to decrease in order to keep the phase constant, so the mode is left-moving.

(b) Converting the solutions $P > 0$ and $P < 0$ in region I yields

$$u_P^R = \begin{cases} e^{-i\omega t} r^{i\frac{\omega}{\kappa}} = e^{i\frac{\omega}{\kappa}\log(-U)} & (P>0) \\ e^{-i\omega t} r^{-i\frac{\omega}{\kappa}} = e^{-i\frac{\omega}{\kappa}\log(V)} & (P<0) \end{cases}, \qquad (U<0, V>0) \tag{20}$$

One notices that since the first solution $(P > 0)$ does not depend on V, it can be trivially extended to $V < 0$. Similarly, since the second solution $(P < 0)$ does not depend on U, it can be extended to $U > 0$, while we can choose to set the waves to 0 in the other regions. Note that since we are dealing with a massless scalar field, these modes represent massless particles moving at the speed of light. These trace 45-degree rays in the Kruskal diagram. The solution $(P > 0)$ represents a massless particle traveling along a right-moving line at 45 degrees, escaping from region III and entering region I. The solution $(P < 0)$ represents a massless particle moving along a 45-degree left-moving line, going from region I to region II. This explains in physical terms how we "extended" the waves from region I to region II/III.

(c) We write the U integral, for $\omega < 0$, as

$$\tilde{f}(\omega) = \int_{\mathbb{R}} dU e^{i\omega U} f(U) = \lim_{R\to\infty}\left(\underbrace{\oint_{C_R} dz e^{i\omega z} f(z)}_{=0} - \int_0^{-\pi} d\theta R i e^{i\theta} e^{i\omega R e^{i\theta}} f(Re^{i\theta})\right) = 0 \tag{21}$$

where C_R is a closed "half-disk" contour encircling the lower-half plane, hence results in zero since we assume the function f is analytic and has no pole. Moreover, assuming $\omega < 0$ the second term has an exponentially suppressed integrand since when the radius of the arc in the lower-half plane goes to infinity, the exponent $i\omega Re^{i\theta}$ gets negative infinite real part for $\theta \in (-\pi, 0)$. The only region which could cause a finite result is the region $\theta \approx 0$ or $\theta \approx \pi$, if f took big enough of values to compensate. However, this region contributes a negligible contribution because of the assumption $\lim_{R\to\infty} f(Re^{i\theta}) = 0$, $(\theta \in (-\pi, 0))$.

(d) We define the logarithm as

$$\log\big(re^{i\theta}\big) = \log r + i\theta\,, \qquad \theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right), r > 0 \tag{22}$$

This creates a branch cut at $\theta = \frac{3\pi}{2}$, and thus $\log(-z)$ is analytic in the lower half z plane. To glue u_R^P, non-zero for negative U, to $\bar{u}_L^R$, non-zero for positive U, let us start by rewriting both parts in terms of $\log(-U)$, which is our desired analytic function that is analytic in the lower half plane. We use

$$\log(-U) = \log\big(Ue^{i\pi}\big) = i\pi + \log U \implies \log U = \log(-U) - i\pi \qquad (U>0) \tag{23}$$

which means

$$\bar{u}_P^L \propto \exp\left(i\frac{\omega}{\kappa}\log U\right) = \exp\left(i\frac{\omega}{\kappa}\log(-U) + \frac{\pi\omega}{\kappa}\right) \qquad (U>0) \tag{24}$$

We get rid of that extra factor by multiplying by $e^{-\frac{\pi\omega}{\kappa}}$, implying that

$$u_P^R + e^{-\frac{\pi\omega}{\kappa}}\bar{u}_P^L = \exp\left(i\frac{\omega}{\kappa}\log(-U)\right) \tag{25}$$

for all $U \in \mathbb{R}$. This function trivially extends analytically in the lower half-plane, given our branch cut choice for the logarithm. It thus has no $e^{-i\omega U}$ contribution with $\omega < 0$. Note also the trivial (important) fact that it has no $e^{-i\omega V}$ contribution since the function is independent of V. Thus, $u_P^R + e^{-\frac{\pi\omega}{\kappa}} \bar{u}_P^L \propto v_P^{(1)}$ is indeed made of modes of positive frequency with respect to Minkowski time T.

One can repeat the same argument for $P < 0$. In that case, the non-trivial check is to verify that the function of V, when expanded into Minkowski modes, has no $e^{-i\omega V}$ contribution with $\omega < 0$, which is equivalent to showing that the function can be extended analytically in the lower half V-plane. Playing the same game as before, we obtain the desired result.

Finally, the same game can be played for the complex conjugated case. For $P > 0$, we have

$$u_P^L = e^{-i\frac{\omega}{\kappa}\log U} = e^{-i\frac{\omega}{\kappa}\log(-U) - \frac{\omega\pi}{\kappa}} \qquad (U > 0) \tag{26}$$

$$\bar{u}_P^R = e^{-i\frac{\omega}{\kappa}\log(-U)} \qquad (U < 0) \tag{27}$$

so the two can be glued by multiplying the second by $e^{-\frac{\omega\pi}{\kappa}}$, yielding

$$u_P^L + e^{-\frac{\pi\omega}{\kappa}} \bar{u}_P^R = e^{-i\frac{\omega}{\kappa}\log(-U) - \frac{\omega\pi}{\kappa}} \tag{28}$$

which is analytic in the lower half U plane. Same game for $P < 0$.

(e) Taking the scalar product and using $(u_P^R, \bar{u}_P^L) = 0$, $(\bar{u}_P^L, \bar{u}_P^L) = (u_P^R, u_P^R)$, we get

$$\begin{aligned} (v_P^{(1)}, v_P^{(1)}) &= |D_P^{(1)}|^2((u_P^R, u_P^R) - e^{-2\frac{\pi\omega}{\kappa}}(\bar{u}_P^L, \bar{u}_P^L)) \\ &= 2|D_P^{(1)}|^2 e^{-\frac{\pi\omega}{\kappa}} \sinh(\pi\omega/\kappa)(u_P^R, u_P^R) \end{aligned} \tag{29}$$

and similarly for $D_P^{(2)}$, hence we fix

$$D_P^{(i)} = \frac{e^{\frac{\pi\omega}{2\kappa}}}{\sqrt{2\sinh(\pi\omega/\kappa)}} \tag{30}$$

At this point we have

$$v_P^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} u_P^R + e^{-\frac{\pi\omega}{2\kappa}} \bar{u}_P^L) \tag{31}$$

$$v_P^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} u_P^L + e^{-\frac{\pi\omega}{2\kappa}} \bar{u}_P^R) \tag{32}$$

Remember that bars denote complex conjugation. After some direct algebra, one gets the desired inverse relation

$$\begin{aligned} u_P^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} v_P^{(1)} - e^{-\frac{\pi\omega}{2\kappa}} \bar{v}_P^{(2)}), \\ u_P^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} v_P^{(2)} - e^{-\frac{\pi\omega}{2\kappa}} \bar{v}_P^{(1)}). \end{aligned} \tag{33}$$

Finally, we project these relations onto $(\cdot, \Phi)$. Note that $a_P^{(i)} \equiv (v_P^{(i)}, \Phi)$ implies, taking the conjugate,

$$a_P^{(i)\dagger} = \overline{(v_P^{(i)}, \Phi)} = (\Phi, v_P^{(i)}) = -(\bar{v}_P^{(i)}, \Phi) \tag{34}$$

where we used $\overline{(\alpha, \beta)} = (\beta, \alpha)$, $(\alpha, \beta) = -(\bar{\beta}, \bar{\alpha})$ and $\bar{\Phi} = \Phi$ since Φ is a real scalar field. Hence, while projecting, we get $\bar{v}_P^{(i)} \to -a^{(i)\dagger}$, with an extra minus sign. This finally gives

$$\begin{aligned} b_P^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} a_P^{(1)} + e^{-\frac{\pi\omega}{2\kappa}} a_P^{(2)\dagger}) \, , \\ b_P^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}}(e^{\frac{\pi\omega}{2\kappa}} a_P^{(2)} + e^{-\frac{\pi\omega}{2\kappa}} a_P^{(1)\dagger}) \, . \end{aligned} \tag{35}$$

as claimed.

3. Radiating Black Holes

(a) $T_H = \frac{1}{8\pi M}$. Reintroducing the units (this can be done uniquely up to a numerical coefficient), we get

$$T_H = \frac{M_{pl}^2 c^2}{8\pi k_b M} = 6 \times 10^{-8}\mathrm{K} \tag{36}$$

The typical wavelength, by Wien's displacement law, is inversely proportional to the temperature

$$\lambda \sim \frac{M_{pl} c^2}{k_b T_H} l_{pl} \sim r_s \approx 3\mathrm{km} \tag{37}$$

which is of the order of the Schwarzschild's radius.

(b) The power radiated by the black hole is

$$P = \sigma T_H^4 \times A = \frac{2\pi^5}{15}\frac{1}{(8\pi)^4} 16\pi \frac{M_{pl}^2}{M^2}\frac{M_{pl}c^2}{t_{pl}} = \frac{q}{M^2} \tag{38}$$

where $q \equiv \frac{\pi^2}{1920}\frac{M_{pl}^3 c^2}{t_{pl}}$. To get an idea of the magnitude, a black hole with the same mass as the sun is characterised by $P \approx 10^{-26}$W.
Hence we have

$$\frac{d(Mc^2)}{dt} = -\frac{q}{M^2} \tag{39}$$

$$t_{evap} = -\int_{M_0}^{0} dM \frac{M^2 c^2}{q} = \frac{M_0^3 c^2}{3q} = \frac{M_0^3}{3M_{pl}^3} t_{pl} \approx 10^{65}\text{years} \tag{40}$$

The power emitted by black holes is very small compared to that of any star (e.g. $\mathcal{O}(10^{26}W)$ for the sun, and it is even lower for large black holes. Moreover, star-sized black holes, which are the smallest known to form from gravitational collapse of supernovae, take extremely long times to evaporate. Finally, the radiation temperature is extremely low even compared to the CMB temperature.
(No wonder Hawking did not win a Nobel prize for this...)

(c) Consider the RN horizons at radii $r_\pm = M \pm \sqrt{M^2 - Q^2}$.
The black hole entropy is $S_{BH} = \pi r_+^2$, while the temperature is (see previous problem

set) $T_H = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{2r_+ - 2M}{4\pi r_+^2}$.
After some straightforward computations, we get:

$$C = T_H \frac{\left.\frac{dS_{BH}}{dr_+}\right|_Q}{\left.\frac{dT_H}{dr_+}\right|_Q} = \frac{2S_{BH}\sqrt{M^2 - Q^2}}{M - 2\sqrt{M^2 - Q^2}} \tag{41}$$

(d) Suppose we slightly increase the temperature of the heat reservoir, then energy will flow into the black hole, increasing its mass. If the heat capacity of the black hole is negative, and therefore as the black hole gains mass, it will decrease in temperature, causing more flux from the reservoir to the black hole. Thus there is a positive feedback effect; the system is unstable.

Now consider 41: the numerator of the heat capacity is always positive, while the denominator changes sign when $M = M_{cr} = \frac{2}{\sqrt{3}}|Q|$.
When $M < M_{cr}$, $C > 0$, hence the black hole is stable. Note that Schwarzschild black holes can never be stable.